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# NOTE

## Note on Geometric Graphs

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segments connecting the corresponding points. We show that a geometric graph of $n$ vertices with no $k+1$ pairwise disjoint edges has at most $2^9k^2n$ edges. © 1999 Academic Press

### 1. INTRODUCTION

A *geometric graph* $G$ is a graph drawn in the plane by (possibly crossing) straight line segments, i.e., it is defined as a pair $G=(V, E)$, where $V$ is a set of points in general position in the plane and $E$ is a set of closed segments whose endpoints belong to $V$.

The following question was raised by Avital and Hanani [AH], Kupitz [K], Erdős and Perles. Determine the smallest number $e_k(n)$ such that any geometric graph with $n$ vertices and $m>e_k(n)$ edges contains $k+1$ pairwise disjoint edges.

It follows from a result of Kupitz [K] that $e_k(n) \geqslant kn$ for any $k \leqslant n/2$. Pach and Törőcsik [PT] proved that $e_k(n) \leqslant k^4 n$ for any fixed $k$, which was the first upper bound linear in $n$. Both the upper and lower bounds were improved by Tóth and Valtr [TV] to $\frac{3}{2}(k-1)\,n-2k^2 \leqslant e_k(n) \leqslant k^3(n+1)$ $(k \leqslant n/2)$. In this note we further improve the upper bound.

THEOREM 1. *For any* $k<n/2$,

$$e_k(n) \leqslant 2^9 k^2 n.$$

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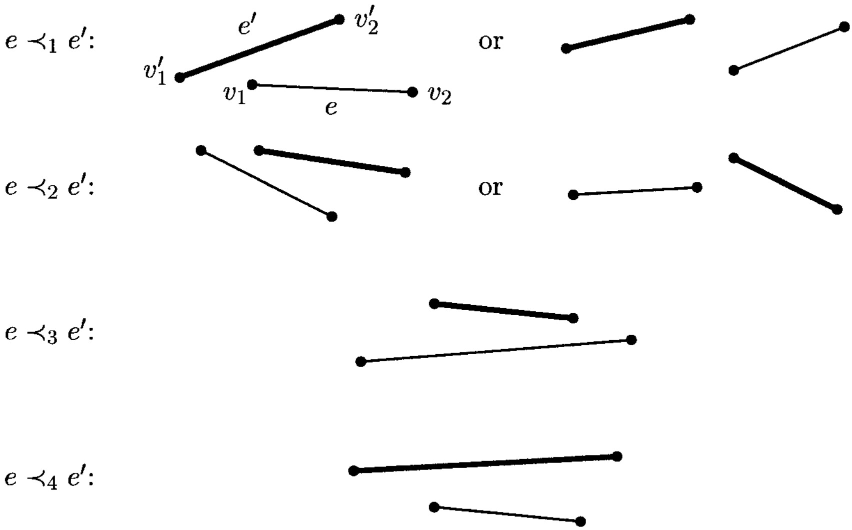


**FIGURE 1**

Let $G$ be a geometric graph. For any vertex $v$, let $x(v)$ and $y(v)$ denote its $x$- and $y$-coordinate, respectively. An edge $e$ is said to *lie below* an edge $e'$, if every vertical line intersecting both $e$ and $e'$ intersects $e$ strictly below $e'$.

Define four binary relations $\prec_i$ $(i = 1, ..., 4)$ on the edge set $E$ as follows (see also [PT, PA, TV]). Let $e = v_1 v_2$, $e' = v_1' v_2'$ be two disjoint edges of $G$, where $x(v_1) < x(v_2)$ and $x(v_1') < x(v_2')$. Then (see Fig. 1)

$$
\begin{aligned}
&e \prec_1 e', && \text{if} \quad x(v_1) \geqslant x(v_1'),\ x(v_2) \geqslant x(v_2'), \text{ and } e \text{ lies below } e',\\
&e \prec_2 e', && \text{if} \quad x(v_1) \leqslant x(v_1'),\ x(v_2) \leqslant x(v_2'), \text{ and } e \text{ lies below } e',\\
&e \prec_3 e', && \text{if} \quad x(v_1) \leqslant x(v_1'),\ x(v_2) \geqslant x(v_2'), \text{ and } e \text{ lies below } e',\\
&e \prec_4 e', && \text{if} \quad x(v_1) \geqslant x(v_1'),\ x(v_2) \leqslant x(v_2'), \text{ and } e \text{ lies below } e'.
\end{aligned}
$$

Each of the relations $\prec_i$ is a partial ordering, and any pair of disjoint edges of $G$ is comparable by at least one of them. Theorem 1 is a direct consequence of the following stronger statement.

THEOREM 2. *Let $k \leqslant n/2$ and let $G$ be a geometric graph with no $k+1$ edges forming a chain with respect to any of the partial orders $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$. Then*

$$e(G) \leqslant 2^9 k^2 n.$$

*For $k = O(\sqrt{n})$ this result can not be improved apart from the value of the constant.*

The relations $\prec_1, \prec_2, \prec_3, \prec_4$ were introduced by Pach and Törőcsik [PT]. In fact, their result was analogous to Theorem 2, with the weaker bound $e \leqslant k^4 n$.

## 2. PROOF OF THEOREM 2

For any graph $G$, let $e(G)$ denote the number of edges of $G$. Let $G$ be a geometric graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and with no $k+1$ edges forming a chain in any of the partial orderings $\prec_1, ..., \prec_4$. If there are two vertices with the same $x$-coordinates, we can perturb them to have different $x$-coordinates. It is easy to see that this way we did not create any additional chain. Therefore, we can suppose without loss of generality that all vertices have different $x$-coordinates and the vertices are numbered from left to right.

For any vertex $v_i$, the *left edges* (resp. *right edges*) of $v_i$ are those edges $v_i v_j$ of $G$, where $i > j$ (resp. $i < j$). The *left degree* $l_i$ (resp. the *right degree* $r_i$) of $v_i$ is the number of left edges (resp. right edges) of $v_i$.

LEMMA. *Let $X = \{x_1, x_2, ..., x_m\}$ be a sequence of different real numbers. Then there are pairwise disjoint monotone subsequences $X_1, X_2, ..., X_l \subset X$ such that for $i = 1, 2, ..., l$, $|X_i| = \lceil \sqrt{m/2}\, \rceil$, and $|X_1| + |X_2| + \cdots + |X_l| \geqslant m/2$.*

*Proof.* Take a monotone subsequence of size $\lceil \sqrt{m/2}\, \rceil$ of $X$ and delete it from $X$. Continue as long as there are at least $m/2$ elements of $X$ left. It can be done by the Erdős-Szekeres Theorem [ES35]. ∎

Return to the proof of Theorem 2. Do the following procedure on $G$, for $i = 1, 2, ..., n$.

RIGHT DECOMPOSITION PROCEDURE $[i]$. Let $v = v_i$, $r = r_i$ and let $e_1, e_2, ..., e_r$ be the right edges of $v$ in clockwise order (such that the clockwise angle enclosed by $e_1$ and $e_r$ is less than $180°$). Let $x(e_j)$ denote the $x$-coordinate of the endpoint of $e_j$ different from $v$. By the Lemma, the sequence $x(e_1), x(e_2), ..., x(e_r)$ contains monotone subsequences, each of size $\lceil \sqrt{r/2}\, \rceil$ such that their total size is at least $r/2$. It defines a partition of the corresponding edges into subsequences. Call each subset of those edges which belong to the same subsequence, *right-block* of edges at $v_i$. Delete those edges which do not belong to any of the subsequences. For any remaining edge $e_j$, we say that the *type* of $e_j$ is *right-increasing* (resp.

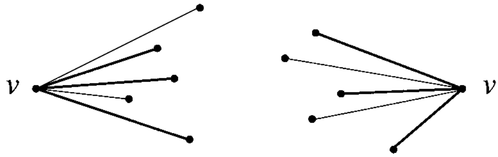


**FIG. 2.** The left graph shows the right-increasing subsequence, the right graph is the left-increasing subsequence.

*right-decreasing*) if $x(e_j)$ belongs to an *increasing* (resp. *decreasing*) subsequence. (See Fig. 2.)

Call the resulting graph $G_1$. Clearly, $e(G_1) \geqslant e(G)/2$. Since every edge of $G_1$ is either of type right-increasing or right-decreasing, at least half of the edges are of the same type, say, right-increasing. (The other case can be treated analogously, as explained in the remark at the end of the paper.) Delete all right-decreasing edges from $G_1$, and call the resulting graph $G_2$. It follows that $e(G_2) \geqslant e(G)/4$.

Let $l'_1, l'_2, ..., l'_n$ be the left degrees of $v_1, v_2, ..., v_n$ respectively, in $G_2$. Since $G_2 \subset G$, $l'_i \leqslant l_i$. Apply the Left Decomposition Procedure on $G_2$, analogous to the Right Decomposition Procedure. Let the resulting graph be $G_3$, we have that $e(G_3) \geqslant e(G)/8$. Suppose that at least half of the edges of $G_3$ are left-increasing. (The other case can be treated analogously, as explained in the remark at the end of the paper.) Delete all left-decreasing edges from $G_3$, and call the resulting graph $G_4$. It follows that $e(G_4) \geqslant e(G)/16$.

For two edges of $G_4$ with a common endpoint, $e_1 = v_i v_j$, $e_2 = v_i v_k$ we say that $e_2$ is a *right-zag* of $e_1$, if both $e_1$ and $e_2$ are right edges of $v_i$, and $e_2$ follows immediately after $e_1$ in the same right-block at $v_i$. Analogously, for $e_1 = v_i v_j$ and $e_2 = v_i v_k$ we say that $e_2$ is a *left-zag* of $e_1$, if both $e_1$ and $e_2$ are left edges of $v_i$, and $e_2$ follows immediately after $e_1$ in the same left-block at $v_i$.

A path $e_1 e_2 \cdots e_m$ of $G_4$ is said to be a *zig-zag path* if one of the following three conditions holds.

(i) $m = 1$

(ii) For any $1 \leqslant i \leqslant m-1$, $e_{i+1}$ is a right-zag of $e_i$ if $i$ is odd and a left-zag if $i$ is even.

(iii) For any $1 \leqslant i \leqslant m-1$, $e_{i+1}$ is a right-zag of $e_i$ if $i$ is even and a left-zag if $i$ is odd.

Observe that each edge of $G_4$ has at most one right-zag and one left-zag. Also, each edge is a right-zag and a left-zag of at most one edge. Therefore, each edge of $G_4$ is contained in at most *two* maximal zig-zag paths.

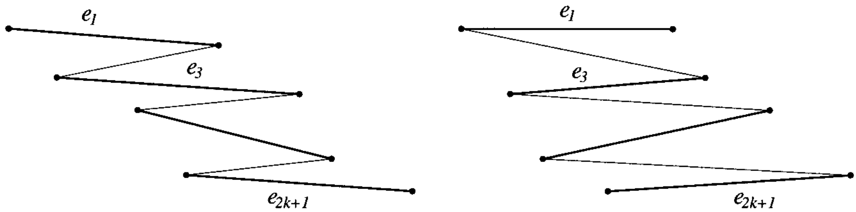


**FIG. 3.** $e_1 \prec_1 e_3 \prec_1 \cdots \prec_1 e_{2k+1}$.

Claim 1. Every zigzag path in $G_4$ has at most $2k$ edges.

*Proof.* Suppose that $e_1 e_2 \cdots e_{2k+1}$ is a zig-zag path and let $1 \leqslant i \leqslant 2k-2$. First we show that $e_i \prec_1 e_{i+2}$. Suppose that $e_i = v_a v_b$, $e_{i+1} = v_b v_c$, and $e_{i+2} = v_c v_d$. We distinguish two cases.

*Case* 1. $e_{i+1}$ is a right-zag of $e_i$ and $e_{i+2}$ is a left-zag of $e_{i+1}$. Then $e_{i+1}$ follows $e_i$ in a right block of $v_b$, so $x(v_a) < x(v_c)$. Also, $e_{i+2}$ follows $e_{i+1}$ in a right block of $v_c$, so $x(v_b) < x(v_d)$. Clearly, $e_{i+2}$ is below $e_i$, so $e_i \prec_1 e_{i+2}$.

*Case* 2. $e_{i+1}$ is a left-zag of $e_i$ and $e_{i+2}$ is a right-zag of $e_{i+1}$. Then $e_{i+1}$ follows $e_i$ in a left block of $v_b$, so $x(v_a) < x(v_c)$. Also, $e_{i+2}$ follows $e_{i+1}$ in a left block of $v_c$, so $x(v_b) < x(v_d)$. Clearly, $e_{i+2}$ is below $e_i$, so $e_i \prec_1 e_{i+2}$.

Consequently, $e_1 \prec_1 e_3 \prec_1 e_5 \prec_1 \cdots \prec_1 e_{2k+1}$ so there is a chain of length $k+1$, a contradiction (see Fig. 3.). This concludes the proof of Claim 1. ∎

Claim 2. There are at most $\sqrt{2e(G)/n}$ maximal zig-zag paths.

*Proof.* For each vertex $v_i$, the number of maximal zig-zag paths starting at $v_i$ is at most the number of blocks of edges at $v_i$. Since each right block in $G$ has size $\lceil \sqrt{r_i/2} \rceil$, the number of right blocks at $v_i$ in $G$ is at most $\sqrt{r_i/2}$. Therefore, the number of right blocks at $v_i$ in $G_4$ is also at most $\sqrt{r_i/2}$. Similarly, the number of left blocks at $v_i$ in $G_2$ is at most $\sqrt{l'_i/2}$, so the number of left blocks at $v_i$ in $G_4$ is at most $\sqrt{l'_i/2} \leqslant \sqrt{l_i/2}$. Therefore, for the total number $Z$ of maximal zig-zag paths in $G_4$ we have that

$$Z \leqslant \sum_{i=1}^{n} \left(\sqrt{r_i/2} + \sqrt{l_i/2}\right) \leqslant \sqrt{n}\, \sqrt{\sum_{i=1}^{n} r_i + l_i} = \sqrt{2e(G)\, n}. \quad ∎$$

Each edge of $G_4$ is covered by at most two maximal zig-zag paths, hence using Claims 1 and 2 we get that

$$e(G_4) \leqslant \frac{1}{2}\, 2k \sqrt{2e(G)\, n}.$$

Therefore,

$$\frac{e(G)}{16} \leqslant \frac{1}{2}\, 2k \sqrt{2e(G)\, n},$$

which implies that

$$e(G) \leqslant 2^9 k^2 n.$$

This concludes the proof of the upper bound. For the lower bound assume that $k \leqslant \sqrt{n/2}$ and consider the following geometric graph $G(k, n)$. Take a slightly perturbed $k \times k$ piece of a unit square grid and rotate it slightly anticlockwise direction. Place the remaining $n - k^2$ points very far to the right and connect each vertex in the lattice with each of the remaining vertices. $G(k, n)$ has $n$ vertices, $k^2 n - k^4 \geqslant k^2 n/2$ edges, and it is easy to see that there are no $k+1$ edges that form a chain with respect to any of the relations $\prec_i$. If $\sqrt{n/2} \leqslant k \leqslant c\sqrt{n}$ then consider $G(k', n)$ with $k' = \sqrt{n/2}$ (suppose for simplicity that it is an integer). $G(k', n)$ has $n$ vertices, $n^2/4 \geqslant k^2 n/4c^2$ edges, and there are no $k+1$ edges that form a chain with respect to any of our relations $\prec_i$. ∎

*Remarks.* (1) It the proof of the upper bound, we assumed that the edges of $G_4$ belong to *right-increasing* and *left-increasing* blocks. It the other three cases the proof is analogous. The only difference is that in the proof of Claim 1 we have to use $\prec_2$, $\prec_3$, or $\prec_4$ in place of $\prec_1$. See Fig. 5.

(2) Theorem 2 guarantees that any geometric graph with $n$ vertices and $e > 2^9 k^2 n$ edges contains $k+1$ edges that form a chain. Following the proof of Theorem 2, it is easy to design a polynomial algorithm that finds such a set of $k+1$ edges.

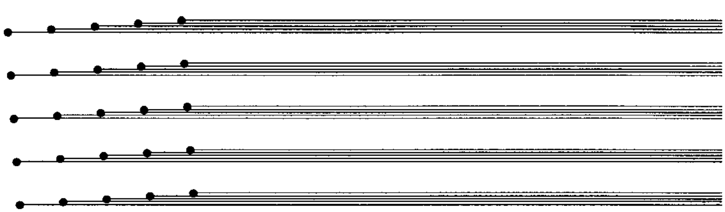

**FIGURE 4**

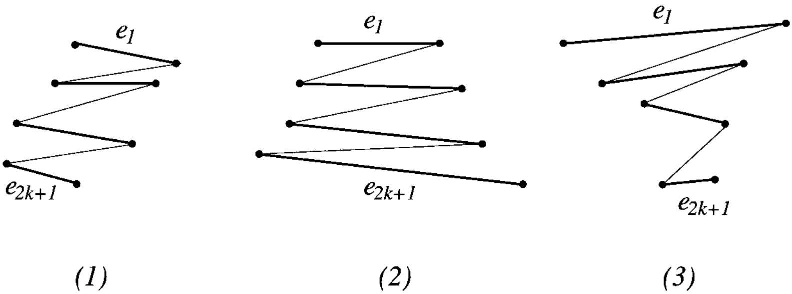


**FIG. 5.** (1) The edges of $G_4$ belong to *right-decreasing* and *left-decreasing* blocks. Then $e_1 \prec_2 e_3 \prec_2 \cdots \prec_2 e_{2k+1}$. (2) The edges of $G_4$ belong to *right-increasing* and *left-decreasing* blocks. Then $e_1 \prec_3 e_3 \prec_3 \cdots \prec_3 e_{2k+1}$. (3) The edges of $G_4$ belong to *right-decreasing* and *left-increasing* blocks. Then $e_1 \prec_4 e_3 \prec_4 \cdots \prec_4 e_{2k+1}$.

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